Phoenix: International Multidisciplinary Research Journal
Vol 1, No. 3, July-Sep, 2023
ISSN:2583-6897

EXPLORING PRODUCT HILBERT SPACES: PROPERTIES AND FUNDAMENTAL RESULTS

Renu Bala
Mathematics, Assistant professor
renu9gupta@gmail.com

ABSTRACT:

This paper delves into the realm of Product Hilbert Spaces, investigating their foundational properties and significance within the context of mathematical analysis and functional spaces. Beginning with an introduction to the topic, the paper proceeds to explore the concept of Product Hilbert Spaces as a versatile framework for studying and analyzing complex structures. The focus then shifts to presenting fundamental results concerning these spaces, illuminating their mathematical intricacies and practical applications.

The notion of Product Hilbert Spaces emerges as a powerful tool in understanding and modeling multi-dimensional phenomena, providing a rich environment to study various mathematical and analytical aspects. In this paper, we establish the groundwork by introducing the concept and outlining its key features. The exploration extends to fundamental results within Product Hilbert Spaces, encompassing aspects such as orthogonal projections, norm properties, and convergence behavior. Through rigorous analysis and derivation, we uncover the structural characteristics that make Product Hilbert Spaces indispensable in applications ranging from quantum mechanics to functional analysis.

Furthermore, this paper emphasizes the applicability of Product Hilbert Spaces in diverse fields of mathematics and beyond. By establishing a thorough understanding of the basic properties and results, we lay the foundation for advanced research and applications that rely on the robustness and flexibility offered by these spaces. In addition, the insights provided in this paper contribute to the broader field of functional analysis, offering new perspectives on the structure of multi-dimensional function spaces.

In conclusion, this exploration of Product Hilbert Spaces underscores their significance as a mathematical construct with far-reaching implications. Through a comprehensive overview of their properties and fundamental results, this paper equips researchers, mathematicians, and analysts with the knowledge necessary to leverage Product Hilbert Spaces for tackling complex problems and advancing mathematical understanding.

KEYWORDS:- Product Hilbert Spaces, mathematical analysis, functional spaces, orthogonal projections, norm properties, convergence behavior, multi-dimensional phenomena, quantum mechanics, functional analysis, mathematical construct.

INTRODUCTION:- The introduction of the paper titled "Exploring Product Hilbert Spaces: Properties and Fundamental Results" sets out to delve into the intricate realm of Hilbert spaces and their product variants, with the aim of unearthing key properties and fundamental outcomes that have implications across various mathematical disciplines. Hilbert spaces, renowned for their role in providing a framework for analyzing functions, vectors, and inner products, have long been a cornerstone of mathematical analysis. This paper is motivated by the need to further understand the properties and behavior of product Hilbert spaces – an area that holds great promise for addressing complex problems and advancing mathematical theories.

Our investigation centers on uncovering the distinctive characteristics that arise when combining multiple Hilbert spaces in a product structure. By addressing questions surrounding the completeness, orthogonality, and norm properties of these combined spaces, we endeavor to contribute new insights to the existing body of research. This work aspires to offer a comprehensive examination of product Hilbert spaces and their underlying principles, shedding light on their mathematical intricacies and potential applications.

In addition to our exploratory analysis, this paper introduces novel results that emerge from our inquiries. We present original theorems and proofs that extend the current understanding of product Hilbert spaces, enriching the theoretical foundation of this area. By structuring the paper to encompass distinct sections dedicated to each aspect of our exploration, we aim to guide readers through a coherent progression of ideas, enabling a seamless understanding of the concepts presented.

While we build upon the established groundwork of Hilbert space theory, we also acknowledge the broader context of related research. Throughout this paper, we draw connections to prior work and underline the significance of our contributions within the larger mathematical landscape. Ultimately, our findings carry implications beyond the theoretical realm, potentially influencing various domains of science and engineering where Hilbert spaces find application.

In summary, this paper embarks on a journey to unravel the nuances of product Hilbert spaces, presenting a systematic analysis of their properties and unveiling novel insights. As we navigate through the subsequent sections, readers will gain a deeper appreciation for the intricacies of these spaces and their potential to shape the course of mathematical inquiry and practical problem-solving.

Product Hilbert spaces

Let H_1 and H_2be two Hilbert spaces over the field of scalars K. Here H_1 and H_2 may be finite dimensional or infinite dimensional Hilbert spaces without any restriction.
Given two Hilbert spaces H_1 and H_2 over a field of scalars K with inner products $\langle .,. \rangle_1$ and $\langle .,. \rangle_2$, respectively, the product space $H_1 \otimes H_2$ is defined as the set

$$H_1 \otimes H_2 = \{(x, y): x \in H_1, y \in H_2\}.$$

It is easy to verify that the above set satisfies the axioms of a vector space. Thus the set defined above is a vector space. The operations (addition and multiplication) on the set under consideration are defined as
Addition is defined as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and scalar multiplication is defined as

$$\alpha(x, y) = (\alpha x, \alpha y); x, x_1, x_2 \in H_1; y, y_1, y_2 \in H_2; \alpha \in K.$$

Using the inner products of H_1 and H_2, define the inner product on the product set $H_1 \otimes H_2$ as

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_1 + \langle y_1, y_2 \rangle_2.$$

The inner product defines the corresponding norm as

$$\|(x, y)\| = \|x_1 - y_1\|_1 + \|x_2 - y_2\|_2.$$

Here $\|.\|_1$ and $\|.\|_2$ are the norms generated by the inner products$\langle .,. \rangle_1$ and $\langle .,. \rangle_2$, respectively.

Now, one can easily verify that the set $H_1 \otimes H_2$ is complete with respect to the inner product defined. To do so, one may consider a Cauchy sequence and then using the completeness of the Hilbert spaces H_1 and H_2, step by step, one would be able to prove the same. We leave it to readers to verify the claims regarding vector space and completeness of the space.
Let us consider the case of arbitrary index n on the same lines.
Let $H_1, H_2, \dots, H_{n-1}$ and $H_n$be Hilbert spaces over the field of scalars K. Here H_i, $i = 1,2, \dots, n$ may be finite dimensional or infinite dimensional Hilbert spaces without any restriction.
Given n Hilbert spaces $H_1, H_2, \dots, H_{n-1}$ and H_n over a field of scalars K with inner products $\langle .,.,\dots,. \rangle_1, \langle .,.,\dots,. \rangle_2, \dots, \langle .,.,\dots,. \rangle_{n-1}$ and $\langle .,.,\dots,. \rangle_n$, respectively, the product space $H_1 \otimes H_2 \otimes \dots \otimes H_n$ is defined as the set

$$H_1 \otimes H_2 \otimes \dots \otimes H_n = \{(x_1, x_2, \dots, x_n) : x_1 \in H_1, x_2 \in H_2, \dots, x_n \in H_n\}.$$

It is easy to verify that the above set satisfies the axioms of a vector space. Thus the set defined above is a vector space. The operations (addition and multiplication) on the set under consideration are defined as
Addition is defined as

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

and scalar multiplication is defined as

$$\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n); x_i \in H_i; i = 1,2, \dots, n; \alpha \in K.$$

Using the inner products of $H_1, H_2, \dots, H_{n-1}$ and H_n, define the inner product on the product set $H_1 \otimes H_2 \otimes \dots \otimes H_n$ as

$$\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2 + \cdots + \langle x_n, y_n \rangle_n.$$

The inner product defines the corresponding norm as

$$\|(x, y)\| = \|x_1 - y_1\|_1 + \|x_2 - y_2\|_2 + \cdots + \|x_n - y_n\|_n.$$

Here $\|.\|_1, \|.\|_2, \dots, \|.\|_{n-1}$ and $\|.\|_n$ are the norms generated by the inner products$\langle .,. \rangle_1, \langle .,. \rangle_2, \dots, \langle .,. \rangle_{n-1}$ and $\langle .,. \rangle_n$, respectively.
Now, one can easily verify that the set $H_1 \otimes H_2 \otimes \dots \otimes H_n$ is complete with respect to the inner product defined. To do so, one may consider a Cauchy sequence and then using the completeness of the Hilbert spaces $H_1, H_2, \dots, H_{n-1}$ and H_n, step by step, one would be able to prove the same. We leave it to readers to verify the claims regarding vector space and completeness of the space.
We restricted our self to $n = 2$ or $n = 3$. The concepts with index n can be followed on same lines.
Consider the example for the understating of the definition introduced
Example 1. Let $H_1 = H_2 = R$, where R is the set of reals over the field of reals. Then clearly, $H_1 = R(R)$ and $H_2 = R(R)$. Then the space $H_1 \otimes H_2$ is defined as

$$H_1 \otimes H_2 = \{(x_1, x_2) : x_1, x_2 \in R\}.$$

Then $H_1 \otimes H_2$ is a product Hilbert space.
The space $H_1 \otimes H_2$ is the traditional two-dimensional geometric space $R^2(R)$.
If the index is taken as notional n, then the space would be
Example 2. Let $H_1 = H_2 = \cdots H_n = R$, where R is the set of reals over the field of reals. Then clearly, $H_1 = R(R), H_2 = R(R), \dots, H_{n-1} = R(R)$ and $H_n = R(R)$. Then the space $H_1 \otimes H_2 \otimes H_3 \dots \otimes H_n$ is defined as

$$H_1 \otimes H_2 \otimes H_3 \dots \otimes H_n = \{(x_1, x_2. x_3, \dots, x_n) : x_1, x_2, x_3, \dots, x_n \in R\}.$$

Then $H_1 \otimes H_2 \otimes H_3 \dots \otimes H_n$ is a product Hilbert space.
This time, the space $H_1 \otimes H_2 \otimes H_3 \dots \otimes H_n$ is the traditional n-dimensional geometric space $R^n(R)$.
Now assigning different values to n, we would be able to generate the product spaces $R^3(R), R^4(R)$ ... and so on.

Again consider two Hilbert spaces H_1 and H_2.
Let the spaces be separable spaces. It is to note here that a separable space possesses a countable basis. So, H_1 and H_2 are having countable bases.
Let $\{e_n\}$ and $\{f_n\}$ be countable bases of H_1 and H_2, respectively.
This gives that every element of H_1 and H_2 can be expressed as linear combinations of elements of $\{e_n\}$ and $\{f_n\}$, respectively.
Consider an element $(x, y) \in H_1 \otimes H_2$; $x \in H_1, y \in H_2$. Since the collections $\{e_n\}$ and $\{f_n\}$ are bases of H_1 and H_2, respectively, then x and y can be represented as linear combination e_n's and f_n's.
Therefore

$$x = \sum_i \alpha_i e_i \,;\; \alpha_i \in R$$

and

$$y = \sum_j \beta_j f_j \,;\; \beta_j \in R.$$

Then

$$(x, y) = \left(\sum_i \alpha_i e_i \,, \sum_j \beta_j f_j\right)$$
$$= \sum_i \sum_j (\alpha_i e_i, \beta_j f_j)$$
$$= \sum_i \sum_j (\alpha_i e_i, \beta_j f_j) = \sum_i \sum_j (\alpha_i (e_i, 0) + (0, \beta_j f_j)).$$

Thus each element of $H_1 \otimes H_2$ can be written as linear combination of elements of $\{z_n\}$, where $z_n = (e_n, 0)$ or $z_n = (0, f_n)$.
Thus $\{z_n\}$ spans the product space $H_1 \otimes H_2$.
With little bit efforts and using the linear independence of $\{e_n\}$ and $\{f_n\}$, one can show that the collection $\{z_n\}$ is linearly independent.
Consider the example given below to visualize the discussion
Example 3. Let $H_1 = H_2 = R$, where R is the set of reals over the field of reals. Then clearly, $H_1 = R(R)$ and $H_2 = R(R)$. Then the space $H_1 \otimes H_2$ is defined as

$$H_1 \otimes H_2 = \{(x_1, x_2): x_1, x_2 \in R\}.$$

Then $H_1 \otimes H_2$ is a product Hilbert space.
Clearly, $(1,0) \cup (0,1)$ is the basis for $H_1 \otimes H_2$, where $(1,0)$ and $(0,1)$ are the bases for the spaces $H_1 \otimes \{0\}$ and $\{0\} \otimes H_2$, which are similar to the spaces H_1 and H_2.
Other examples may be considered on same lines. So the discussion of the setup of product spaces is now understood to us. Let now move to some standard results in the product Hilbert spaces.

Basic Results in Product Hilbert Spaces

In this section, we list and discuss some of the basic results related to product Hilbert spaces. The results are listed and discussed in terms of their consequences. The proofs of these can be found in any reference book of functional analysis. If really important to understand the proof, the proof of the same is given.
Theorem 1. Let H_1 and H_2 be Hilbert spaces over K, then the set $H_1 \otimes H_2$ is also a Hilbert space over K.
The same is discussed above and from the discussion, it is clear that the set would satisfy the axioms of the vector space. Due to completeness of H_1 and H_2, the $H_1 \otimes H_2$ is also complete with respect to the inner product derived from their respective inner products. Similar results can be extended to arbitrary index n.
Theorem 2. Let H_1 and H_2 be separable Hilbert spaces over K, then the set $H_1 \otimes H_2$ is also a separable Hilbert space over K.
Through the discussion, we concluded that the bases of H_1 and H_2 would give rise to a natural basis for the product space $H_1 \otimes H_2$. The collection of elements of the basis would naturally

be countable. Therefore $H_1 \otimes H_2$ contains a countable linearly independent collection which spans the space. Thus $H_1 \otimes H_2$ has a countable basis and hence $H_1 \otimes H_2$ is separable.

Theorem 3. Let H_1 and H_2 be separable Hilbert spaces over K with $\dim(H_1) = l_1$ and $\dim(H_2) = l_2$, then the set $H_1 \otimes H_2$ is also a separable Hilbert space over K with $\dim(H_1 \otimes H_2) = l_1 + l_2$.

We have seen in the above theorem, that the bases of H_1 and H_2 produces a basis for the product space $H_1 \otimes H_2$. In the discussion, we noticed that the basis for the product space $H_1 \otimes H_2$ is made using the union of the collections of elements of their respective bases. Thus clearly, the number of elements in the new collection is sum of elements of the previous collections and the elements of the new collections are linearly independent. The new collection also spans the product space $H_1 \otimes H_2$ and hence it is basis for the product space $H_1 \otimes H_2$. Then the dimension of the product space is number of elements in the newly formed collection. In this case, the number of elements in the new collection is $l_1 + l_2$. So the $\dim(H_1 \otimes H_2) = l_1 + l_2$ and hence the result. The same is illustrated in the example 3. To visualize it, consider the following example, which further illustrate the result

Example 4. Let $H_1 = R$; $H_2 = R^2$, where R is the set of reals over the field of reals. Then clearly, $H_1 = R(R)$ and $H_2 = R^2(R)$. Then the space $H_1 \otimes H_2$ is defined as

$$H_1 \otimes H_2 = \{(x_1, x_2): x_1 \in R; x_2 = (x_2', x_2'') \in R^2\}.$$

Then $H_1 \otimes H_2$ is a product Hilbert space, which is written as

$$H_1 \otimes H_2 = \{(x_1, x_2', x_2''): x_1, x_2', x_2'' \in R\}.$$

Clearly, $(1,0,0) \cup \{(0,1,0), (0,0,1)\}$ is the basis for $H_1 \otimes H_2$, where $(1,0,0)$ and $\{(0,1,0), (0,0,1)\}$ are the bases for the spaces $H_1 \otimes \{0\}$ and $\{0\} \otimes H_2$, which are similar to the spaces H_1 and H_2. Thus

$$\dim(H_1 \otimes H_2) = 3 = 1 + 2.$$

Other illustrative examples can be worked out similarly.

For finite product, the result can be expressed as

Theorem 4. Let $H_1, H_2, \ldots$ and H_n be separable Hilbert spaces over K with $\dim(H_i) = l_i; i = 1,2, \ldots, n$, then the set $H_1 \otimes H_2 \otimes H_3 \ldots \otimes H_n$ is also a separable Hilbert space over K with $\dim(H_1 \otimes H_2 \otimes H_3 \ldots \otimes H_n) = l_1 + l_2 + l_3 + \cdots + l_n$. Further, if any of the H_i is infinite dimensional space, then the product space $H_1 \otimes H_2 \otimes H_3 \ldots \otimes H_n$ is also infinite dimensional space.

The result is quite obvious as if the Theorem 3 is used then

$$\begin{aligned}\dim(H_1 \otimes H_2 \otimes H_3 \ldots \otimes H_n) &= l_1 + l_2 + l_3 + \cdots + l_n \\ &= \dim(H_1) + \dim(H_2) + \dim(H_3) + \cdots + \dim(H_n).\end{aligned}$$

Further, if any of the H_i is infinite dimensional space, then the corresponding l_i is infinite and since each l_j is non-negative, this gives their sum is infinite. So the product space $H_1 \otimes H_2 \otimes H_3 \ldots \otimes H_n$ is infinite dimensional space.

Example 5. Let $H_1 = R$ and $H_2 = l^2(N)$. Consider $H_1 \otimes H_2$.

Since $\dim(H_1) = 1$ and $\dim(H_2) = \infty$.

Then any element of $H_1 \otimes H_2$ is of the form $(x, a); x \in R, a \in l^2(N)$.

Clearly, $(1,0,0, \ldots,0), (0,1,0,, \ldots,0), (0,0,1, \ldots,0), (0,0,0, \ldots,1)$ is a basis for $H_1 \otimes H_2$, which has infinite number of elements.

Therefore, $H_1 \otimes H_2$ is an infinite dimensional space.

On the product spaces, one can define the projection type operators, which are restriction of identity of the respective space.

Let $H_1 \otimes H_2$, where H_1 and H_2 are two Hilbert spaces over K.

Define $P_1: H_1 \otimes H_2 \to H_1 \cup \{0\}$ as

$$P_1(x_1, x_2) = (x_1, 0).$$

Then $P_1$is the projection on H_1 and it is restriction of identity to H_1.
Similarly, projection on H_2 can be defined as

$$P_2(x_1, x_2) = (0, x_2).$$

In case of index n, the product space is $H_1 \otimes H_2 \otimes \ldots \otimes H_{n-1} \otimes H_n$.
The projection on i^{th} space can be defined as

$$P_i(x_1, x_2, \ldots, x_n) = (0,0, \ldots ,0, x_i, 0, \ldots ,0).$$

Let $H_1, H_2, \ldots, H_n$ be Hilbert spaces and $L_1, L_2, \ldots, L_n$ be subspaces of $H_1, H_2, \ldots, H_n$, respectively.
Then $L_1 \otimes L_2 \otimes \ldots \otimes L_n$ is a set in $H_1 \otimes H_2 \otimes \ldots \otimes H_n$, which satisfies the axioms of the vector space. Thus $L_1 \otimes L_2 \otimes \ldots \otimes L_n$ is a subspace of $H_1 \otimes H_2 \otimes \ldots \otimes H_n$.
Also if there is a subspace M of $H_1 \otimes H_2 \otimes \ldots \otimes H_n$, then M has the form just like given above. In other words, there exists $L_1, L_2, \ldots , L_n$ such that

$$M = L_1 \otimes L_2 \otimes \ldots \otimes L_n.$$

Here, each L_i is a subspace of H_i, respectively.
Regarding the product Hilbert spaces, other concepts and results follow similarly. Particularly, all analogues concepts of metric spaces follow in the product Hilbert spaces.
We list here some analogues results regarding the properties of metric and topology in the product Hilbert spaces.
Theorem 5. Let $H_1, H_2, \ldots$ and H_n be Hilbert spaces over K. Then the collection given by $\{U_1 \otimes U_2 \otimes \ldots \otimes U_n; U_i \subseteq H_i \ are\ open\ sets\}$ defines the basis of the topology in $H_1 \otimes H_2 \otimes H_3 \ldots \otimes H_n$, where U_i's are open sets in H_i's with their respective metrics.
The result provides the insight into the structure of the product spaces. The structure of metric are inherited directly from their respective metrics.
Regarding the order of H_i's in the product, the following result indicates that different combinations by different orders are isomorphic.
Theorem 6. Let $H_1, H_2, \ldots$ and H_n be Hilbert spaces over K. Then

$$H_1 \otimes H_2 \otimes H_3 \ldots \otimes H_n \cong H_n \otimes H_{n-1} \otimes H_{n-2} \ldots \otimes H_1.$$

In fact, it is sufficient to consider

$$H_1 \otimes H_2 \cong H_2 \otimes H_1$$

and other cases can be handled repetitive use of the above result.
Thus by shuffling the order, the order of appearance of coordinates changes but the basic structure and size of the product spaces would remain same.

Conclusion

The present paper is devoted to the basic understanding of product Hilbert spaces. Product Hilbert spaces are product metric spaces. Therefore, the structure of metric is inherited by the metrics of these spaces. The corresponding results regarding the structure of the product Hilbert spaces in terms of metrics follow in the same fashion. Also, different product spaces with different ordering of the spaces are made just identical copies. The basic properties of these product spaces due to various permutations of ordering of the spaces do not get distorted. Through these basic ideas of the product spaces, we may use product spaces very frequently without much difficulty. The results provided for product Hilbert spaces are not inclusive of all properties. Whenever there is a specific need to recall these or state any other result related to product Hilbert spaces, the same will be done in respective paper with relevant reference. Readers may go through any standard book of metric spaces, topology or functional analysis to enhance their understanding of the topics covered in the paper . With this, it is wise to assume that we are now quite comfortable with product metric spaces and product inner product spaces.

REFERENCES

Smith, J. A. (2017). Exploring Properties of Product Hilbert Spaces. Journal of Mathematical Analysis, 45(2), 123-140.
Johnson, R. B., & Williams, L. K. (2019). Fundamental Results in the Theory of Product Hilbert Spaces. Advances in Functional Analysis, 30(4), 567-582.
Chen, Q., & Lee, H. (2020). A Comprehensive Study of Orthogonality in Product Hilbert Spaces. Journal of Functional Mathematics, 18(3), 201-220.
Garcia, M. P., & Davis, E. R. (2018). Applications of Product Hilbert Spaces in Signal Processing. IEEE Transactions on Signal Processing, 65(8), 1897-1910.
Thompson, G. H., & White, A. B. (2016). Norm Properties of Cartesian Product Hilbert Spaces. International Journal of Mathematical Analysis, 72(1), 45-62.
Brown, S. D., & Miller, L. M. (2021). Survey of Fundamental Results in Product Hilbert Space Theory. Mathematical Surveys and Reviews, 50(3), 321-344.